Part II of my course

In this part of the course, I wanted to start from ~~particles in~~ solving the quantum mechanics problem for both non-relativistic and relativistic particles in an EM field. Then I will go on to give a brief idea about how it is related to the phenomenon of the quantum Hall effect in semi-conductors as well as in graphene.

Let us start with the problem of a spinless single particle in a magnetic field. The Hamiltonian for the (NR) system is given by

$$H = \frac{1}{2m} \Sigma \left(\vec{p} - \frac{e\vec{A}}{c} \right)^2$$

We shall take the magnetic field to be uniform and in the $\hat{z}$-direction. As you probably know, there are several gauges in which one can solve the problem. Let us choose the Landau gauge given by

$$A_x = -B_0 y \quad , \quad A_y = 0 \; , \; A_z = 0$$

~~In this gauge~~

$$\vec{\nabla} \times \vec{A} = \vec{B} = B_0 \hat{z}$$

as required because $B_z = \hat{z}\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right)$

Now, let us expand the Hamiltonian

$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{e}{2mc}\left(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}\right) + \frac{e^2 \vec{A}^2}{2mc^2}$$

According to the rule for commutation of mtm operator with any fn. of the co-ordinates

$$\vec{p}\cdot\vec{A} - \vec{A}\cdot\vec{p} = -i\hbar\, \vec{\nabla}\cdot\vec{A}$$

So if $\vec{\nabla}\cdot\vec{A} = 0$, then they commute.

Here, in the gauge we have chosen $\vec{\nabla}\cdot\vec{A} = 0$, so we can be careless about the ordering.

So we can write

$$H = \frac{1}{2m}\left[\left(p_x + \frac{eBy}{c}\right)^2 + p_y^2 + p_z^2\right]\psi$$

$$= E\psi$$

It is easy to see that the solution

has to be of the form

$$\psi(x,y,z) = e^{ip_z z} e^{ip_x x} \chi(y)$$

since there is no potential in either the x or z - direction. So the eigenvalues p_z and p_x can take all values from $-\infty$ to $+\infty$.

In terms of $\chi(y)$, the eq^{n} becomes

$$\frac{\partial^2 \chi}{\partial y^2} + \frac{2m}{\hbar^2}\left[\left(E - \frac{p_x^2}{2m}\right) - \frac{1}{2} m\omega^2 (y-y_0)^2\right] = 0$$

where $\omega = \dfrac{eB_0}{mc}$ and $y_0 = -\dfrac{cp_x}{eB_0}$

We have chosen to re-arrange the eq^{n} that we get by naively substituting to be written in this form, because it is easy to see that the Hamiltonian in y is just that of a shifted harmonic oscillator.

So the energy eigenvalues can be directly written down

$$E_{n,p_x,p_z} = \left(n+\tfrac{1}{2}\right)\hbar\omega + \frac{p_z^2}{2m}$$

and the eigenfns are

$$\psi(x,y,z) = \frac{1}{(\text{Norm})} e^{\frac{ip_x x}{\hbar}} e^{\frac{ip_z z}{\hbar}} e^{-\frac{(y-y_0)^2}{2l^2}} H_n\left(\frac{y-y_0}{l}\right)$$

Hermite polynomials

where $l = \sqrt{\frac{\hbar}{m\omega}} = \sqrt{\frac{\hbar c}{eB}}$

What are the things to notice about ~~these~~ these solutions. ?

~~In this gauge, the sol^n looks free in the~~

Let us see how the sol^{ns} look in the x-y plane.

Classically, you know that the motion of particles in the plane $\perp^r$ to $\vec{B}$ is given as circular orbits.

Now, we have solved the problem quantum mechanically. What does it look like ?

It looks like the ptcle is free to move in the x-direction, but it falls off exponentially around y_0 in the y-direction with a length scale $\sim l$.

The other point to note, is that there is no dependence at all of the energy E on the p_x quantum #.
The energy remains the same for all values of p_x — which means that there is a huge degeneracy in the problem.
For each value of n, the energy level called Landau level is infinitely degenerate.
We can actually calculate the degeneracy by considering a finite size for the system. Let us consider just the planar system and consider a rectangular box of $L \times W$.

So the p_x values are quantised as $\frac{2\pi\hbar n_x}{L}$

And the centre of the shifted h.o. y_0 can lie between 0 & W.

So $0 < y_0 < W$

Now we had earlier defined $y_0 = -\frac{cp_x}{eB_0}$

$= \frac{2\pi n_x}{L} \frac{\hbar c}{eB_0} = \frac{2\pi n_x}{L} l^2$

$\therefore \frac{2\pi n_x}{L} \frac{\hbar c}{eB_0} < W$

$\therefore \frac{n_x}{WL} < \frac{eB_0}{2\pi\hbar c} = \frac{eB_0}{hc}$

So # of allowed levels per unit area $= \frac{eB_0}{hc}$ = degeneracy of each Landau level. $\frac{hc}{e}$ is often called the unit of flux $= \Phi_0$

Hence degeneracy of Landau levels is also given by $\frac{B}{\Phi_0}$ or the # of fluxes per unit area or flux density.

Before I go on to explain how this quantum mechanical problem is related to the phenomenon of the quantum Hall effect, let me also briefly mention how the solution looks in different gauges because it is quite illustrative.

The same problem can also be done in the circular gauge

$$A_x = -\frac{By}{2} \qquad A_y = \frac{Bx}{2}, \qquad A_z = 0$$

(also satisfies $\vec{\nabla}\cdot\vec{A} = 0$)

In this case, we can work with (r, θ, z) co-ordinates since the problem has circular (or cylindrical symmetry).

But it is also convenient to use the $z, \bar{z}$ co-ordinates (which we will later comes useful when we think about the quantum Hall problem.

So, let us define

$$z = \frac{x+iy}{\ell} \qquad \bar{z} = \frac{x-iy}{\ell} \qquad = \sqrt{\frac{\hbar c}{eB_0}}$$

($z, \bar{z}$ dimensionless ℓ = length parameter

Then

$$\partial \equiv \partial_z = \ell(\partial_x - i\partial_y) \qquad \bar{\partial} \equiv \partial_{\bar{z}} = \ell(\partial_x + i\partial_y)$$

It is an excercise to rewrite the Hamiltonian as

$$H = \frac{\hbar\omega}{2}\left(-\bar{\partial}\partial + \bar{z}z - z\partial - \bar{z}\bar{\partial}\right)$$

The simplest way now to solve this problem is to define raising & lowering operators

$$a = \frac{\bar{\partial} + z}{\sqrt{2}}, \qquad a^{+} = -\frac{\partial + \bar{z}}{\sqrt{2}}$$

and you can check that

$$[a, a^{+}] = 1, \quad [a, a] = 0, \quad [a^{+}, a^{+}] = 0.$$

Then we get

$$H = \hbar\omega\left(a^{+}a + \tfrac{1}{2}\right)$$
$$= \left(n + \tfrac{1}{2}\right)\hbar\omega$$

How do we see the ∞ degeneracy of the Landau levels here?

Note that a particle in 2 dimensions has 2 degrees of freedom (2 pos & 2 mtm)

So you actually need 2 more operators, which are independent of the a & a^+ to represent the particle. They are

$$b = \frac{\partial + \bar{z}}{\sqrt{2}} \quad \& \quad b^+ = \frac{-\bar{\partial} + z}{\sqrt{2}}$$

and these are also ladder operators with

$$[b, b^+] = 1 \quad , \quad [b, b] = 0 = [b^+, b^+]$$

Also $\quad [a, b] = [a, b^+] = 0$

It happens that the energy is independent of these b & b^+ operators (just as in the other gauge it happened to be independent of p_x). So there is an ∞ degeneracy in the system.

To understand the physical meaning of these operators, note that a, a^+ increase (or decrease) the energy. So one can go from one Landau level to another. Hence, the radius of the orbits increases (this can also be seen explicitly if we go to the (r, θ) co-ordinate. So different eigen functions correspond to different raddi of orbits. But the oscillators, b, b^+ correspond to shifting of the centre of the orbits. It is also possible to do the degeneracy counting here by considering a finite area, but I won't do that here.

The only point that I want to emphasize is that there is a degeneracy in the Landau levels and the wave-fns can look very different in different gauges.

~~To~~ We can easily get the ~~waf~~ wave-fns in this gauge. using the H.O - wfns

Let us call the wfns $\phi_{n,m}(z, \bar{z})$

n → q# of a^+; m → q.# of b^+

Grnd state is given by

$$a\,\phi_{0,0} = b\,\phi_{0,0} = 0$$

Solving for this, we can get

$$\phi_{00} = \underset{\hookrightarrow \text{const.}}{D_{00}}\, e^{-|z|^2}$$

And then we can get all the other states degenerate with this state by acting b^+ on this state

$$\phi_{0,m} = (b^+)^m \phi_{00}$$
$$= D_{0m}\, z^m e^{-|z|^2}$$

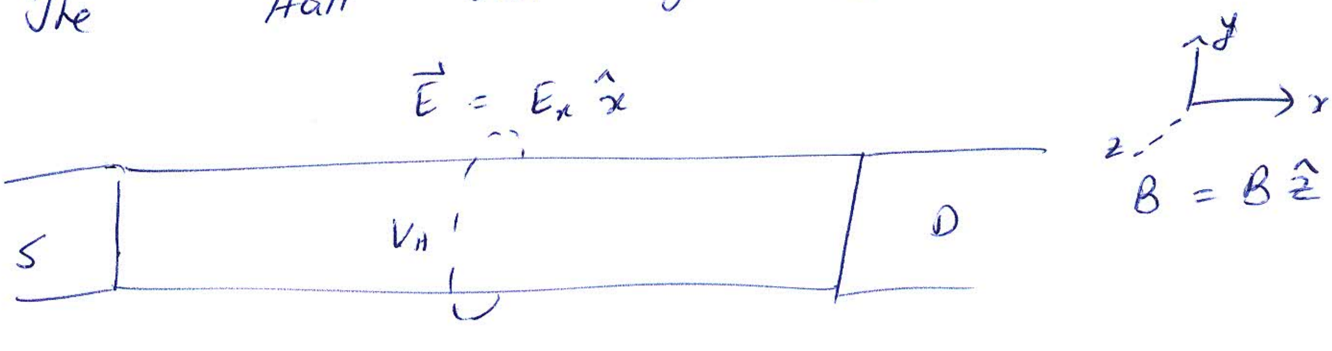
Since the Hamiltonian is independent of b, b^+, all these states are degenerate with $\phi_{0,0}$.

But if you plot their wave-fns, you will find that they correspond to radial patterns with increasing r for larger m.

To go to higher Landau levels, one needs to ~~include~~ act by $(a^+)^n$. In this case, the energy will increase. Also $\bar{z}$ factors will appear. As you may have noted earlier, as long as we acted by (b^+), the w-fn only had z-factors – i.e. it was analytic. This is one of the features of the lowest Landau levels.

Module of QHE

How does all this relate to the physical phenomenon of the quantum Hall effect? Many of you may have forgotten about the quantum Hall geometry, so let me just remind you about it. The Hall bar geometry is given by



The Hall resistance R_H is defined as

$$R_H = \frac{V_H}{I} = \frac{E_y W}{j_x W} = \frac{E_y}{j_x}$$

$$= \rho_{xy}$$

Since the geometric width cancels out, B the Hall resistance is the same as the Hall resistivity.

If ~~you~~ we calculate it ~~tot~~ classically using

$$\vec{F} = m\dot{\vec{v}} = (-e\vec{E}) + (-e)\,\vec{v} \times \frac{\vec{B}}{c}\,,$$

$$j = ne\vec{v}\,,$$

we ~~get~~ find that

$$\sigma_{ij} = \begin{pmatrix} 0 & -\frac{nec}{B_0} \\ \frac{nec}{B} & 0 \end{pmatrix} \quad \text{or}$$

$$\rho_{xy} = \begin{pmatrix} 0 & \frac{B_0}{nec} \\ -\frac{B_0}{nec} & 0 \end{pmatrix}$$

showing that the off-diagonal ~~conductance~~ resistance was proportional to B_0.

But instead when it was measured, ~~it~~ the conductance σ_{xy} was found to be quantised in units of e^2/h and exactly at these values of $1/B_0$. the diagonal resistivity was found to be zero. But this was in a real materials and so on. This quantisation, in other words, occurs in a complex macroscopic system, independent of material details, sample geometry etc, independent of temperature and disorder. The accuracy was 5 parts per million in the original experiment itself. Now, it is the IQHE which is used as the std of electrical resistance.

A many particle effect is being used as a standard!

Later, at lower temperatures, and in ~~clanes~~ cleaner samples they found the FQHE. But that depends on e-e interactions. I will not talk about that here. Instead, I will restrict myself to IQHE, its relationship to other topological insulators and IQHE in graphene.

Now, how do we understand the IQHE from the energy levels of a single particle in a $\perp$ mag. field?

While doing the QM problem, we defined $\phi_0 = \frac{hc}{e}$ as the unit of flux. We also said that the degeneracy of the Landau levels is given by B/ϕ_0 which is the amount of flux per unit area of the sample or the flux density.

Now, let us define something called the filling factor ν. ν is defined as the ratio of the electron density to the flux density. So

$$\nu = \frac{n}{B/\phi_0} = \frac{n\phi_0}{B} = \frac{nhc}{eB}$$

So $$\rho_{xy} = \frac{B}{nec} = \frac{1}{\nu} \cdot \frac{h}{e^2}$$

or $$\sigma_{xy} = \frac{\nu e^2}{h}$$

It is ν which is quantised as an integer in the IQHE (and as a fraction with odd denominator in the FQHE).

So what we now see ~~th~~ is that the plateaux ~~now~~ are exactly at those values where a certain # of Landau levels are completely filled.

How exactly does this explain the plateaux? Well, this needs a little more work.
Very naively, one can argue that filled Landau levels implies stability because any excitation requires energy to overcome the Landau level gap of ω_c, which is quite large.
But how exactly does this lead to plateaux?